MATHEMATICAL EXPLORATIONS AND ‘TALK’ AT THE MARGINS

Synopsis of PhD Thesis

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by

Jayasree Subramanian

Advised by
Prof K Subramaniam & Prof R Ramanujam

Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research
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1 Overview

The popular belief about mathematics is that it is meant for a few with “mathematical minds” and beyond the vast majority of people. As a school subject it is perhaps considered the most difficult, leading to failure and school dropouts. On the one hand it is empowering in that it opens up opportunities for well-paying jobs and upward social mobility. On the other hand the value that our society accords to it leads to anxiety and the sense of shame in non-achievers. The marginalising power of mathematics and its tendency to function as a “social filter” is well acknowledged in literature. When it comes to socio-economically marginalised students, whose educability itself is sometimes questioned, mathematics often functions as a barrier to finishing school. The central concern of this study is ways of mitigating the marginalising effects of mathematics especially for those students who are already marginalised due to their socio-economic and educational backgrounds and “recentering the margins”.

Literature identifies the narrow concerns of school mathematics for the one-right answer sanctioned by the authority of the textbook, the privileging of repeated practice to master specified problem types and concepts, and normatively defined ways of communicating mathematics as marginalising. Moving away from this we sought to design and implement tasks that enable a rich mathematical experience even in marginalised or low resource contexts. We started with flexibility and accessibility as key design principles guiding task design and identified task features that enable flexibility and accessibility. Explorations that are low threshold and high ceiling, allowing for multiple starting points, approaches and trajectories and affordances to function at multiple levels of formalisation are tasks that align with these design principles. We therefore sought to study the potential of explorations to recentre the margins.

Following a first-person-classroom-based approach to research, we facilitated and observed students as they engaged with mathematical explorations. We observed students engaging in such practices as literature identifies as elements of mathematical thinking - conjecturing and convincing, specialising and generalising, representing and rerepresenting, imagining and expressing. We also noted the ways in which they express their mathematical thinking and the resources that they draw on. We noted the prevalence of oral communication in informal language and the near absence of symbolisation and formalisation as distinctive features that mark their engagement with such tasks. Deviating from the deficit perspectives that fail to acknowledge the mathematical in such conversations, we sought to define more accommodating acceptability criteria for what constitutes mathematical discourse. Keeping in view the key considerations of mathematical practice namely formalisation and consistency, we suggest coherent formalisability (CF) as such an acceptability criterion. We also mark desirable student behaviour associated with the flexible pedagogy that explorations and CF entail, such as attempts at sense making rather than blindly following procedures, assuming agency and aspiring for excellence.

Additionally, we look at what it implies for the teacher to enable flexibility without compromising on core disciplinary constraints. We identify challenges and suggest workarounds in the form of guidemaps and offer a framework for such guidemaps.

The thesis is organised into 8 chapters. The first chapter discusses the marginalising aspects of mathematics and how these lead to a deficit perspective. The different dimensions of marginalisation, namely performative, disciplinary and language, are discussed and the research objectives spelt out.

The second chapter looks at the theoretical constructs that we draw on from literature. We look at the the different ways Mathematical Thinking and Mathematical Discourse have been conceptualised and interpreted in literature. We also look at the role of informal talk in doing mathematics, both in teaching-learning mathematics and in the work of research mathematicians, and literature around open tasks.

The third chapter is on methodology and spells out the research questions, methodological stance, describes the study context, the methods of data collection and analysis, and data sources. It also gives an overview of the explorations that form part of the analysis.

In the next four chapters the findings from the study are discussed. Chapter 4 identifies the task design features that enable flexibility and accessibility. Chapter 5 describes what engagement with mathematical exploration looks like at the margins, looks at the role of talk in this and suggests an acceptability criterion for mathematical discourse in such contexts. Chapter 6 looks at the implications for teachers and chapter 7 at what mathematical engagement ‘could’ look like in a curricular context.

The last chapter discusses the limitations of this study and gives some pointers to future work and implications.

2 Introduction

While policy makers consider education to create opportunities for upward social mobility, critical education researchers point out that schools themselves can act as agents of marginalisation. A curriculum that does not take into account individual student strengths and needs, education characterised by rigid systems and structures, and standards (minimum learning levels in India) driven programmes which create winners and losers, all tend to marginalise some learners. It is widely acknowledged that school mathematics can also marginalise students (Ewing, 2002; Gates & Noyes, 2020; Warren & Miller, 2016). Several studies indicate large differences between mathematical performance of the dominant groups and the marginalised groups (Akmal & Pritchett, 2021; Borooah, 2012). Skovsmose (2011) refers to the simultaneous empowering and disempowering capacity of mathematics - on the one hand, mathematics empowers by opening opportunities for lucrative jobs. On the other hand, the debilitating anxiety about failure in mathematics and the associated shame can be disempowering. Mathematics has also been referred to as a “social filter” (Zevenbergen, 2002), as access to mathematics is mediated by class- and culture-based language use, with students who have difficulty assimilating the socially legitimised linguistic practices being labelled “failures”. We identify three dimensions of marginalisation specific to mathematics:

- The performative dimension stemming from the importance accorded to mathematics in the society (D’Souza, 2021; Ernest, 2020; Luttenberger et al., 2022; Noyes, 2009; Schoenfeld, 2002)
- The disciplinary dimension stemming from what is generally accepted as ways of doing mathematics especially in school context (Ernest, 2003; Lampert, 1990; Sarangapani, 2020; Skovsmose, 2022; Skovsmose & Penteado, 2015; Solomon, 2008)
- The language dimension stemming from what is considered as acceptable ways of talking/communicating mathematics (Barwell et al., 2016; Pimm, 1987; Schleppegrell, 2007; Setati, 2008; Zevenbergen, 2002)

Policy makers, teachers, educators and a section of mainstream research tend to take a deficit view of learners located at the margins (Adiredja & Louie, 2020; L. P. Davis & Museus, 2019; Russell et al., 2022). Such views focus on students’ academic shortcomings without recognising their existing understandings and strengths. They also locate the source of academic problems in deficiencies within students, their families and communities without examining how student outcomes are shaped by school practices and systemic conditions. The pervasiveness of such deficit discourses in the society exerts a pernicious influence even on teachers who consciously wish to counter them and reinforces the myth that children from disadvantaged backgrounds are not educable. Scholars have recognised the need to address and move beyond these deficit perspectives.

The focus of this study is on ways of addressing deficit perspectives and “recentering”what we called the disciplinary dimension and the language dimensions of the margins. Recognising inherently marginalising factors such as the narrow concerns of school mathematics about the one-right answer pre-determined by an authority like the textbook or teacher, the “normative” solution methods based on standard algorithms, and formal mathematical discourse, which have been described by Skovsmose (2001) as constituting “the exercise paradigm”, we sought to create alternate ways for students to engage with mathematics. Skovsmose proposes landscapes of investigation as a learning environment different from the school maths tradition and exercise paradigm.

2 Literature review

2.1 Open Tasks in mathematics education iterature

Central to the landscapes of investigation is that they invite students to engage with the inquiry processes, to frame questions of interest to follow, make visible a solution approach and allow for open-ended conversations. Tasks focussed on developing mathematical thinking as opposed to tasks intended as practice for procedural skills has been the subject of much discussion in mathematics education literature. Scholars use different terms to refer to such tasks: investigative tasks, guided-discovery tasks, problem-posing tasks, open-ended tasks and ill-structured tasks (Yeo, 2007). While these categories are not separated by clear demarcating lines, some common features of such tasks are that they have open goals allowing different students to pursue different goals and are generative of further questions. Ernest (1984) differentiates investigatory tasks, guided discovery tasks and problem solving tasks on the basis of the roles of the teacher and student in each of them. Becker & Shimada (1997) study the potential of open-ended tasks in teaching-learning of what they call higher-order thinking and in evaluating student achievement of this objective in the Japanese context. Jaworski (1994) addresses, among other questions, what does an investigational or enquiry classroom look like, in the UK setting. She characterises an investigatory approach to teaching through the three elements - management of learning, sensitivity to students and mathematical challenge. However there is scant literature on what an investigatory classroom looks like in a low resource context, and the potential of an investigatory approach to mitigate the rigidity of the textbook in such contexts. This study aims to address this gap.

2.2 Elements of mathematical thinking

There are many different definitions and interpretations of the term *mathematical thinking*. Several scholars have identified and put together practices, habits of mind or processes that constitute *doing mathematics* (Bass, 2011; Bell, 1976; Burton, 1984; Cuoco et al., 1996; Mason et al., 1982; Ramanujam, 2010; Watson, 2008). Making conjectures, building on answers to generate new questions, coming up with and choosing between representations, looking for invariances, looking for counterexamples, seeking relationships, observing extreme cases, experimentation, reasoning and generalisation are practices which have been highlighted by multiple scholars.

Drawing a clear distinction between mathematical thinking and the body of knowledge (content and techniques) described as mathematics, Burton (1984) suggests that teaching mathematics content like algebra or geometry or trigonometry compulsorily and over years does not necessarily provide the conditions through which students develop their mathematical thinking. She argues that mathematical thinking is not thinking about the subject matter of mathematics, but “a style of thinking that is a function of particular operations, processes and dynamics, recognisably mathematical.” (p 35). Differentiating developing mathematical thinking from generating problem solving approaches in the classroom, she suggests that making the processes overt and concentrating on them so that they become the focus of the learner’s attention is key to the former. Mathematics (content) is presented as a closed manipulation of techniques, whereas mathematical thinking demonstrates open inquiry. She illustrates this difference by drawing on a routine-sounding percentage problem and describes a solution procedure that starts with an intuitive guess, which is revised based on trying out a specific example, a revised conjecture made, verified and justified and the problem itself generalised. She suggests that an overly diligent focus on the content of mathematics may be expected to obstruct the development of awareness on which mathematical thinking is based. Burton (1984) proposes a model of mathematical thinking in terms of operations, processes and dynamics. She identifies the following as elements of mathematical thinking

- Operations: Enumeration, iteration, study of relationships and transformation
- Processes: Specialising, generalising, conjecturing and convincing
- Dynamics: Manipulating, getting a sense of pattern, articulating the pattern symbolically
- An articulated pattern can be manipulated again and the cyclic process continues with increasing levels of complexity.

In addition to these Mason et.al (1982) also point to the processes of Imagining and expressing, stressing and ignoring, extending and restricting and classifying and characterising and the themes of doing and undoing, invariance in the midst of change and freedom and constraint as markers of mathematical thinking. We draw on Burton’s framework to discuss mathematical thinking.

2.3 Mathematics and language

The language of mathematics with its specialised vocabulary and language structures, compression of meaning achieved through use of symbols and further encapsulation of a series of processes into symbolic expressions, is acknowledged to be alienating and an entry barrier to mathematics. In a country like India, most students having to learn mathematics in a language that is not their home language with some of these being privileged for political, social or economic reasons adds an additional layer to the problem. Scholars have written about the tensions and dilemmas inherent in teaching and learning mathematics in contexts of language diversity.

The tension, arising in contexts where the language of instruction in school and the languages used by students outside school are different, often due to the differential status accorded to languages by the society, presents itself to a teacher as the “dilemma of code-mixing” (Adler, 2002). According primacy to access to mathematics (epistemological access) calls for use of home languages as language of teaching and learning. This however limits access to the “language of power” and the access to social goods that it enables (Setati, 2008). Even where the home language and the language of instruction is the same, navigating between the informal language used in everyday conversations, the academic language used in school or textbook contexts and the disciplinary language which is specific to mathematics creates difficulties. Use of informal language aids sense-making whereas competence to communicate with the larger community of mathematics calls for formal language. This opens up decision points for teachers - whether they should pay explicit attention to mathematical language or leave it implicit so as not to disrupt the mathematical discussion, listen to learners’ exploratory talk and assist the negotiation and development of meaning rather than prematurely offer language help with ways of speaking mathematically (Adler, 2002). This tension between the formal and informal has been observed in several studies in different parts of the world (Adler, 2002; Barwell, 2016; Farrugia, 2013; Khisty, 1995; Moschkovich, 2008).

Looking beyond acknowledging the tension, scholars have also studied the relation between mathematical language and informal language and examined the role of informal language in understanding mathematical concepts. Informal language has been seen to function both as a source of support and a source of interference in understanding mathematical concepts. The overlap of mathematical terminology and everyday words have been observed to be a source of interference. Cornu (1991) investigates the different meanings that students have for the word “limit” . Most often it is considered as an ‘impassable limit’ but it can also mean an impassable limit which is reachable; a point which one approaches, without reaching it; a maximum or a minimum; the end; the finish etc. Students’ use of the mathematical term “limit” is conditioned by these everyday meanings. Scholars have also documented ways in which students and teachers use everyday language to understand mathematical concepts and the nature and purpose for which they draw on informal or mathematical language. Barwell (2012) observes that in classrooms where multiple languages are used, the formal mathematical terms are presented in the official language. This phenomenon has been attested to by other scholars as well (Bose & Choudhury, 2010; Setati, 2005). Moreover Barwell observed that English used in mathematics lessons in Pakistan quoted the textbook, either through reading it aloud or through repetition of the textbook content. Students' informal discussion of mathematical ideas were in the regional language. Setati (2005) goes further to state that in primary school mathematics in South Africa mathematics in English tended to be more procedural in nature, while discussions of students' thinking or mathematical ideas were more likely to be in their home languages.

This aligns with Thom’s (1973) remark as quoted by Burton (1987) on the preference of research mathematicians to resort to informal means in their creative processes rooted, as they are, in meaning making.

> “In practice a mathematician’s thought is never a formalised one… one accedes to absolute rigour only by eliminating meaning; absolute rigour is only possible in, and by, such destitution of meaning. But if one must choose between rigour and meaning, I shall unhesitatingly choose the latter.”

The role of the informal in the processes of mathematical discovery has been acknowledged by others as well (Hadamard, 1945; Müller-Hill, 2013). Barwell (2013) analyses the use of informal language by mathematicians in a radio talk to draw attention to the “mathematical ways” of using everyday language. Radford (2000) and Rowland (2000) draw attention to linguistic means that students use to speak mathematically using informal language. So what makes the talk mathematical is not the language used, but the “ways” in which language is used. What then characterises the “mathematical ways” of using language?

From the perspective of a teacher whose goal is to promote mathematical thinking, it is necessary to balance the systemic need for students to develop the “accepted” ways of communicating mathematics and her goal of having her students discuss rich mathematics. In the process she makes a trade-off in allowing a certain degree of flexibility to express themselves informally while also gently pushing toward more formal discourses. There is no neat resolution to the tension between informal and mathematical language. Insisting on the formalised mathematical language will exclude and disenfranchise many learners who find the formal language of mathematics forbidding. On the other hand, not providing them with the opportunity to learn more formal ways of communicating mathematics, will also in the long run disenfranchise them even if they have a good understanding of mathematics. The teacher’s response to the dilemma may in turn have an influence on students’ participation in mathematics, leading to a vicious circle. We intend to examine how this dilemma presents itself in a low resource context and offer insights that will help the teacher consciously come up with alternate responses to the evolving situation, other than what she is habituated to.

3 Methodology

The deficit perspective on mathematics education sees maths ability as innate and that some /many students *cannot* be good at it. This also leads to a pedagogy that delivers mathematics as cut-and-dried “truths” to be memorised and procedures to be replicated. We wondered if students, especially those at the margins, could be given a different mathematical experience and if yes, what this would involve and how students would respond to this experience. We saw mathematically rich and accessible tasks that enable a flexible pedagogy as key to such experience. Drawing on literature that suggested “landscapes of investigations” as a learning environment that differs from the school maths tradition and the Open University/Association of Teachers of Mathematics (ATM) ideas of explorations, we intended to study the potential of explorations to support mathematical thinking, especially at the margins. By margins, we mean the ‘mathematical margins’ along the dimensions as described in Section 2, comprising of students whose mathematics “achievement” is not what is expected at their grade level, who have not had any prior mathematics experience other than the ‘school mathematics tradition’ and/or those who are not conversant with the language of mathematics. Zevenbergen (2002) draws attention to the influence of a students’ socio-cultural backround and the class-mediated language resources they have access to on their mathematics performance. Thus, there are large overlaps between the socio-economic margins and the mathematical margins.

The research goal of studying the potential of mathematical explorations to support mathematical thinking at the margins, calls for observation of marginalised students engaging with mathematical explorations. We needed to observe students as they engage with mathematical explorations and understand the kind of thinking they engage in when not constrained by curricular goals, their ways of communicating their thinking, the mathematical and linguistic hurdles to engagement and communication, the multiple resources including linguistic resources that they draw on to overcome these hurdles. While we anticipated that the flexibility offered by explorations could be an enabler for mathematical thinking, we also felt the need for constraints so that flexibility does not contravene considerations fundamental to the discipline. This implies that we redefine boundaries for what counts as mathematical thinking and mathematical discourse such that they balance disciplinary considerations and the need for flexibility. We were also sensitive to the constraining factors like the demands on the teacher, prior knowledge requirements both for students and for the teacher and other pragmatic constraints in making explorations a part of the schooling experience of a student and aimed to give pointers for support and/or workarounds. We frame our research questions as

1. What task-features support mathematical thinking at the margins?
2. What does engagement with mathematical explorations entail, at the margins?
    a. What is the nature of mathematical thinking seen in these contexts?
    b. How do students communicate their mathematical thinking?
    c. How does language support or hinder mathematical communication?
    d. What counts as mathematical discourse in such contexts?
3. What does it entail for the teacher to facilitate mathematical explorations at the margins balancing the need for flexibility and the need to adhere to disciplinary considerations?

4. What does mathematical engagement look like in curricular contexts?

We adopt the methodological stance of classroom-based-research (Kelly & Lesh, 2000) from what Ball (2000) calls “researcher-teacher first person perspective”, with elements of design research. The focus of studies in this paradigm is on “what is possible'' rather than in “what is typical” in ordinary classrooms, and the goal is to develop descriptions of existing situations, or conjectures about possible situations. The research results are “existence proofs” or designs of alternate learning environments.

Though the idea of explorations itself is not uncommon, instances of marginalised students engaging with explorations for a long enough period for us to undertake an in-depth study of the phenomenon is rare if not absent in the Indian context. The curricular imperatives and the consequent time-constraints for the regular teacher; and the prevalent deficit perspectives around what these students can or cannot do, make it improbable to find a classroom where the students are engaging with mathematical explorations in anything more than a one-off instance. We therefore had to create the phenomenon we wanted to study (Ball, 2000). Our research collaboration which includes a teacher-researcher with prior teaching experience, a mathematician who also brings in a rich experience of interacting and facilitating explorations with students across levels, and an educational researcher, is ideally positioned to design such a context and undertake a “first-person inquiry’ into teaching to understand the pedagogical elements involved in enabling such a context.

We designed a number of explorations, and the teacher-researcher amongst us facilitated explorations in two schools, School 1 and School 2, catering to students from socio-economically disadvantaged backgrounds, on a weekly basis for close to two years. The students qualify for fee-exemption and mid-day meals and most of them are first generation school-goers. This implies that there is almost no academic support available at home, should they face difficulties in school-learning. Tamil is the first language of most students in the class and language of teacher-student and student-student conversations. The students who were part of this study had English as the medium of instruction. This means that their textbooks were in English, but the regular teaching at school happened in Tamil, along with the Tamil medium students. The sessions on explorations were conducted after school hours and were pitched as an optional enrichment program. The students were assigned to the program based on their interest in mathematics - both self expressed and as judged by their regular teacher.

We base our analysis on the designing and teaching of 7 exploratory modules described in Table 1. The modules were discussed within the research team prior to implementation and course corrections made as they were implemented in class. The teaching sessions were done in the presence of an observer who took notes, and were audio recorded using 2 recorders placed at vantage points in the group. I also wrote reflection notes on most of these sessions. In addition to teaching in these schools, I also facilitated explorations at other venues like a third school, talent nurture camps and summer camps. These instances were not done with the research agenda in mind, nor recorded, however reflective notes were written and the learnings from them discussed within the research team. Though not part of the analysis, these learnings informed the study.

Table 1: Explorations Done

Exploration	Details
Matchstick geometry	Students explore matchstick shapes with focus on the concepts of similarity, congruences, constructable shapes, shapes that can be replicated without measurements, etc.
Guess the colour	A game based module where a 5 x 5 square grid, is divided into two rectangles (horizontally or vertically) and each coloured differently, and students guess the division by asking an optimum number of questions. The problem is extendable and generalisable.
Corner Sums	A puzzle that involves writing numbers along the perimeter of polygons such that the sum of numbers along each side remains the same. The starting point is a triangle with three numbers per side.
Views of Solids	A set of tasks involving visualising, building and sketching solids when one or more of its views are given and identifying incompatible views.
Clapping game	A game based module that is presented as m students sit in a circle, every nth student claps in turn. The point of investigation is for what values of m and n do all students get to clap, and for what values are some students left out? The task branches out to modular arithmetic and star polygons.
Leap Frogs	Another game based module, where the primary task is to interchange a set of black and white tokens arranged in a straight line with a gap of one space in between, following specified rules of movement.
Polygons	In this task students figure out the maximum number of right angles possible in a polygon. Task variation is achieved by interpretations of “polygon” and “right angle”.

We use the teaching in School 1 spanning three academic years 2017- 2018, 2018 - 2019, and 2019 - 2020 and three cohorts as our primary data source.

Table 2: Teaching Details

	Academic year	**Approx number**	**Class**	**Remarks**
Cohort 1	2017 - 2018	15	Class 9	
Cohort 2	2018 - 2019	20	Around 7-8 class 8 students & 12 - 13 class 9 students	The class 8 students here continued to be part of cohort 3
Cohort 3	2019 - 2020	20	Class 9	

The details of these cohorts are as in the table. At the school's request, for both cohorts 2 and 3, I taught a few sessions to help students get a deeper understanding of concepts dealt with in the regular school curriculum. The number of exploratory and curricular sessions I taught across these years and the data sources available are as shown in the table below.

Table 3: Data Sources

	Number of sessions	Audio recorded	Teacher diary written	Observer notes available
Explorations analysed	23 sessions	18 sessions (approx 15.5 hrs)	14 sessions	4 sessions
Other Explorations	11 sessions	2 sessions (approx 2 hrs)	11 sessions	3 sessions
Curricular sessions	17 sessions	5 sessions (approx 4 hours)	15 sessions	6 sessions

In addition, I also helped the class 9 students of cohort 2 along with a few others (a group of 20 in all) with their preparation for the class 10 exams on a weekly basis in the year 2019 - 2020. Thus I interacted with the entire cohort 2 for a second year as well, and had an opportunity to observe them engage with explorations and with curricular mathematics. This extended period of interaction and engagement in teaching beyond the intended research goals, gave us insights that informed the study. A more detailed picture of the data collected and choice of data for analysis will be discussed in the thesis.

We now look at the findings from the study. We first focus on task design and those task features we identified that supported mathematical thinking.

4 Findings

4.1 Task features that support mathematical thinking at the margins

Given our aim of a) creating a learning environment that moves away from the familiar routines of the mathematics classroom; and from the primacy of the textbook and the inherent one-right-answer focus; and b) designing tasks in which every student could make at least some progress and feel a sense of achievement, we identified *flexibility* and *accessibility* as key design principles that support mathematical thinking at the margins. We then ask
i) What task features afford flexibility in tasks?
ii) What task features make them more accessible for students at the margins?

4.1.1 We suggest that the following task features afford flexibility in tasks:
a) Open task formulations that allow room for interpretation and choice of goals to pursue: Textbook problems are generally well specified, containing all the information required to solve the problem and no more. Yeo (2017) discusses multiple axes along which a task could be termed "open", namely i) the goal ii) the method to be adopted iii) the answer that is deemed acceptable iv) possible extensions and v) complexity. However Yeo (2008) suggests that secondary school students who had no prior experience with open investigatory tasks did not know how to start when presented with such a task. Thus a framing where at least one possible goal is spelt out leaving sufficient room for variation may be preferable. *We suggest that an open formulation of the task with partly-specified goals afford flexibility, while also pointing to potential directions of inquiry.* We illustrate the movement towards a more flexible formulation through the example of the evolution of one of our modules, Guess the Colour.

In the initial conceptualisation of the module, the main task was framed as

"Given a 5 x 5 grid of squares, divided into two rectangles of two different colours, say blue (B) and green (G)

a) The colours of how many grid squares would you need to ask for to figure out the colouring of the entire grid?
b) What is the minimum possible number of grid squares that may be revealed so that the entire grid is cracked?

This was followed by a series of guided questions, where the colours of some grid-cells were revealed and students were asked to figure out the colours of as many cells as they could. For example in figure 1(a) from the information given it can be inferred that the entire rectangle with the pair of diagonally opposite corners marked by the Bs will also have to be coloured blue. From figure 1(b) it can be inferred that the square has been vertically split into a 5 x 2 rectangle coloured blue and a 5 x 3 rectangle coloured green.

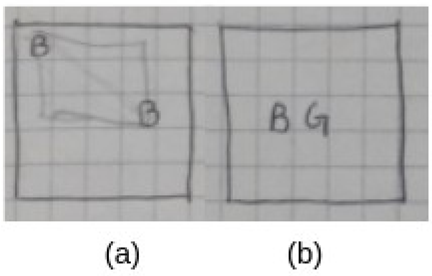

Figure 1: Guess the Colour - Initial version

The expectation was that by engaging with a number of questions like this, students would figure out the combination of cells to ask for that would reveal the maximum information, thereby solving the problem. Two points that we wish to draw attention to in this formulation is that

a) The formulation allows students to ask for only one kind of question - namely what colour is a particular grid-cell. Students would also need to precisely specify the cell whose colouring they wanted to know - for example the cell in the third row, fourth column etc.
b) From inferring additional information based on what is given (as in figure 1) to figuring out the combination of grid-cells that will reveal the entire colouring is perhaps a leap, which might make the task inaccessible at least to some students. More importantly it privileges one solution method and the sub tasks “funnel” towards this privileged method. In that sense it is goal-directed in a manner similar to a textbook problem.

Based on discussions within the research team, the initial formulation was changed to

I have a 5 x 5 grid. I have divided into two rectangles and coloured each with a different colour - One rectangle is red and the other is blue. You have to guess how exactly I have coloured the grid. You can ask me questions.

The kind of questions which could be asked were left open. It was decided to observe what kinds of questions students ask and how they refine the questions if they had to do it in fewer turns. Some of the initial questions that were asked
How many cells have you coloured red?
Is the red 15 upside or downside?

Does the first row have blue?
Have you divided horizontally or vertically?
A “Yes” to the question “does the first row have blue?” could come from a horizontal division with blue at the top of the grid, or a vertical division with the first row having both blue and red cells. To choose between the two possibilities, they came up with a “is there red in the last row?”
And “is blue there in the first row on the left side?” and eventually to the pair of questions “How many blues in the first row and how many blues in the first column?” In later runs of the module elsewhere, more complex and language intensive questions like, questions like “Is the length of the blue rectangle along the top/bottom/left or right edge of the square?” or “is the number of cells in either of the rectangles a multiple of 3?” came up.

Based on the variety of questions that came up and the refinement that happened both in terms of sharpening the question to extract relevant information and refining language, we concluded that the revised and more “open-beginninged” formulation was more in line with our goals than the former. The refining of the questions itself called for mathematical thinking. After a few runs of the game, we chose to restrict to questions that can be answered with a yes/no. We felt that the “how many” questions was giving the game away too soon and that this restriction would keep the challenge level high. Narrowing down the scope of questions also enables better planning for further progress of the exploration and offers an obvious path to generalisation than when any kind of question is admissible. Thus, for this module, an approach that did not overly specify the kind of admissible questions, giving students room to work with their own interpretations of what could be asked, and narrowing down questions as the game progressed depending on the context gave flexibility to the task, without compromising on accessibility and being generative of further questions. This has been corroborated by our experience with other modules as well.

We also identify the following features as bringing about flexibility in tasks. We will discuss the relevant examples in the thesis.

b) Affordances to function at multiple levels of formalisation: It is widely accepted that the symbolic representations and the formalism of mathematics are entry barriers to the discipline. Affordances to work at different levels of formalisation is a feature that we looked for to make the task more flexible and allow students to make some progress, even when functioning at informal or semi-formal levels. For example, game based problem formulations allow for solutions within the same context while demanding mathematical thinking.

c) Incorporating multiple trajectories: Choosing tasks that can branch out along multiple paths possibly to multiple content domains of mathematics is another way of bringing in flexibility. Potential to extend the task by varying the task parameters is a related task feature that opens up multiple trajectories for exploration. In a game based exploration, one could vary the initial configuration, allowed rules for movement or the desired end configuration. Other tasks may give room for such questions as whether a property that is true for a restricted class (e.g. regular polygon) may hold for a larger class (e.g. any convex polygon, any polygon) and if it does not how may one modify (weaken) the statement so that it holds for a larger class etc - This gives students choice to pursue a trajectory that they find interesting and accessible.

4.1.2 We suggest that the following task features make tasks more accessible for students at the margins.

a) Limiting prerequisites: A key consideration while designing tasks has been to minimise dependence on specialised prerequisite knowledge or algorithms to get started on the task. We have observed that at times even what may be considered "grade-appropriate content knowledge" or theorems that students are supposed to have learnt as part of their curriculum can prove to be stumbling blocks to progress when an exploration crucially depends on it. When such a dependence is unavoidable, we sought to incorporate workarounds depending on what the students know and the nature of the result in question, either by having students arriving at the result, or having them "looking up" the result or may be even sharing the result as something that they can take for granted and work with.

b) *Using physical material:* Some tasks can be made more accessible by starting with a hands-on activity using physical material. The question itself may be framed in terms of the activity and may evolve to being framed in more abstract/general terms. The transition from a formulation in terms of physical material to more abstract formulations also creates affordances to work with multiple representations.

c) *Multiple entry points, answers or approaches:* In the example of the Guess the Colour module discussed above, the formulation which does not specify the kind of admissible questions, allows for multiple approaches to the problem. Also there is no one set of questions that need to be asked to elicit the required information. There are multiple ways of doing this. Students are likely to find out one way or another to solve the problem, making it more accessible.

Explorations incorporate most of these features and enable flexibility and accessibility in the mathematics classroom. We now look at what engagement with mathematical explorations entails at the margins.

4.2 What does engagement with mathematical explorations entail, at the margins?

The general perception is that explorations are meant to challenge the mathematically inclined and that explorations cannot be sustained at the margins. The "grade appropriate content knowledge", the academic / formal language in which mathematical thinking is expressed in school context or the learning that comes from prior engagement with mathematical explorations or exposure to other mathematical experiences may be lacking in these contexts. For these reasons the student engagement with mathematical explorations and their articulation of mathematical thinking may look different from what one would expect from students from better- resourced contexts. We now go on to describe the nature of mathematical thinking seen in these contexts, on how students express their mathematical thinking, the resources available to them and how they support or hinder engagement. We use an instance from our module on matchstick geometry as our anchor to discuss these points. We describe the relevant instance and in the subsequent sections draw attention to the features we wish to highlight

The instance we describe here is one where students were trying to replicate the following figure 2 using matchsticks.

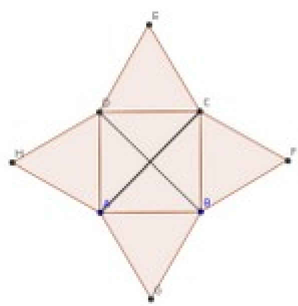

Figure 2: Task from Matchstick Geometry

They realised that they cannot place matchsticks along the diagonal of the square without gaps or overlaps.

Trying to accommodate two sticks along the diagonal, they said

1. "*Ithu edamme paththamatenguthu*" (There is not enough space at all).

With one stick they said

2. *"ithu romba chinnatha irukke, intha end ukku varamatteguthu"* (this is too small, it doesn't come to this end) which *the teacher revoiced as "corner to corner varallaye"* (It is not coming from corner to corner).

One student suggested that they make a bigger square (with more than one matchstick per side). The teacher encouraged them to go ahead and explore this. In the meanwhile another student said that it is not possible to fit the diagonal whatever the size of the square, and the teacher asked to be convinced.

In the following turns of conversation, S1, S2, S3 and S4 are students, J is the teacher, and T is their regular teacher who was observing. Translation into English is given in parentheses.

3. S1: *diagonal vanthu eppovme..*(The diagonal of the square is always…
4. S2: *Square-oda* diagonal is not equal to the side of length, *squarennudayathu* (The diagonal of the square is not equal to the side-length)
5. S1: *ithodu lengthum athodu lengthum equal aa ve irrukkathu* (The length of this and this will not at all be equal
6. S1 & S2: *Diagonal eppovme vanthu,* diagonal is greater than the side of the square (Diagonal is always, diagonal is greater than the side of the square
7. J: How much greater?
8. S1: *Eh? How much aa*? (What? How much?)

…

9. T: you tell me how much bigger, M and K?
10. S1:Double
11. J: *Double aa? Appo rendu kuchci vacha varannume?* (Double? then two sticks should fit)
12. S1: half double.
13. J: *half double na?* (Half double means?)
14. S1: One and a half

…

15. S2: Miss, Pythagoras Theorem. This square plus this square is equal to this square
16. S3: *Hey, yaar sollithantha?* (Hey, Who taught you?)
17. S2: *nangalle kandupidichom* (We found out ourselves)
18. J: Ok, So you are telling me that you cannot make this?
19. S2: *huh huh* (No No)
20. S4: *Pythagorus theorem thane? (* It is Pythagorus theorem isn't it?
21. T: *Appadiya? (*Is it so?)
22. S2: This square plus this square is equal to this length
23. T!: *Enn?* Why?
24. S2: Because it is a right angle. Yes, *Squarekku* right angle sir. (A square has a right angle)
    (Someone claps)

In the meanwhile, the group who was making a larger square fitted a three-unit diagonal to a 2-unit sided square. The teacher drew attention to this and asked the group S1, S2, S3 and S4 if they wanted to reconsider their stand that a matchstick diagonal cannot be fitted to any square. S2 and others pointed out to S1 that she was wrong. The diagonal could be longer than the sides and yet it may be possible to fit in a matchstick diagonal. There was confusion - Some of them sensed something was wrong, but did not connect it to Pythagoras theorem that was mentioned earlier and tried to figure out what was wrong. One student felt that the

2-unit square with the 3-unit diagonal may be flawed (“not perfect”) in some way and wanted to measure the lengths. The teacher suggested that the lengths are obvious - the matchsticks along any side could be counted. A boy pointed out the possibility of the matchsticks being slightly different in lengths and wanted to check the exactness of lengths through a pencil and paper construction. In the meanwhile a group of girls writing on the floor as shown below (figure 3) drew on Pythagoras Theorem again to justify that the square was indeed “flawed” - Pythagoras theorem gives the value of 4 + 4 = 8 for the square of diagonal, whereas the matchstick square on the floor has the value 9.

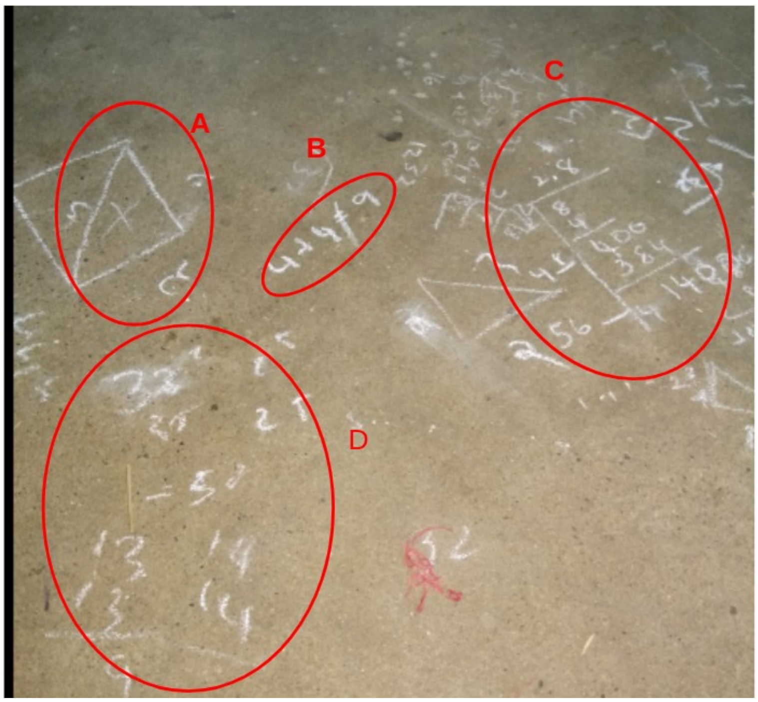

Figure 3: Student work on classroom floor

In order to see if there could be other squares with integer sides and integer diagonal lengths, they looked for perfect squares which when added to themselves would give another perfect square. They tried out specific examples and evaluated the square roots by long division method. At this point the teacher intervened to suggest using factorisation and writing the square root as a surd in a√2 form instead of evaluating it as a decimal. This led to the conclusion that the diagonal is an irrational multiple of the side.

4.2.1 What is the nature of mathematical thinking seen in these contexts?

We now highlight the elements of mathematical thinking in the instance described above.

The question under consideration was whether the given shape as in figure 2 could be constructed with matchsticks. From a content perspective, this is a one step answer - that this is not possible because the diagonal of a square is an irrational multiple of its side. However our interest is in making overt the processes (Burton, 1984) that the group of students went through to arrive at this answer. To arrive at this conclusion, the students went through the process of considering a special case of unit square, making a conjecture for all squares based on the case of unit square, verifying the conjecture for larger squares, encountering an observation that ran counter to their intuitions, investigating the counter-intuitive case and finally arriving at a deductive argument to explain the anomaly leading to a further generalisation of the argument.

All the three elements of Burton's framework for mathematical thinking - operations, processes and dynamics - are evident here. The students here are engaged in the operation of studying relationships - the focus here is the relationship between the side-length and the length of the diagonal of a square. Also evident are the processes of conjecturing, convincing, specialising and generalising. The students also go through the dynamics of manipulating matchsticks, getting a sense of pattern and articulating it. The articulation of the pattern however is not formalised and they do not move to further manipulating symbolically articulated patterns. The initial articulation of Pythagoras theorem is - "this square plus this square is equal to this square". Even when the symbolic articulation of the theorem comes up, it is stated as a memorised fact, and further manipulation of the statement in its general form to argue the case for a general square does not happen. Symbolisation is limited and this is the key difference we mark in how mathematical thinking manifests at the margins.

4.2.2 How do students communicate their mathematical thinking ?

We begin by drawing attention to the following easily noticeable features of the exchange:

1) *Code-mixing:* The language was a mix of Tamil and English, both that of the teachers and the students. The mathematical terms like square, diagonal, double were retained in English and the rest was a mixture of Tamil and English sometimes within the same sentence. There was a tendency to state the "general truths" like diagonals are longer than sides, etc., in English, perhaps from the perception of these statements as theorems. Sentences were often incomplete and not correctly formed.
2) *Informal expression:* The conversation was informal, with students using mathematical vocabulary like square, diagonal, etc., and avoiding symbolism. They frequently drew on the everyday register to express themselves - for example, the idea of the diagonal being greater than 1 and less than 2 is expressed in terms of "space (*idam*)" to accommodate matchsticks. They also fumbled for words to express themselves and came

up with self-created words. If the diagonal is not “double” the side-length, they offered the alternative “half-double” to express the ratio.

3) *Not sufficiently precise:* Precision which one would expect as a distinguishing feature of mathematical discourse is conspicuous by absence, The statement of the Pythagoras theorem was about a particular triangle, pointing to its sides and not as a general statement. Even when they used a more formal articulation later on, as a^2 + b^2 = c^2, what the a, b, c stands for was missing from the articulation. There was ambiguity whether the statement should be interpreted as a relation between the sides of a rectangle and its diagonal that is true only for some rectangles, or whether the reference was to cases where all three are integers.
4) *Interaction drawing on multiple modes, talk being the preferred mode:* Students drew on multiple modes to communicate - speech, writing, diagrams, gestures, etc. There was very little formalised writing seen - the communication was predominantly through talk. Most of the writing was on impermanent surfaces, like the classroom floor, desks or the blackboard though they were requested to write in the notebooks provided. The few pages of writing that were collected over the year were more in the nature of scribbles made to aid the thought process, rather than that meant to communicate thought to a wider audience. A sample of the kind of writing seen is in Figure 3. The impossibility of the diagonal of length 3 is expressed by just a cross in the appropriate place in the image that they drew to aid thinking about the problem (part A in image above) and an oral statement to the effect that the diagonal is wrong. That Pythagoras theorem is violated was expressed through the inequality $4 + 4 \neq 9$ (part B in the figure). There was no accompanying text that explained what this is supposed to signify. The significance of the statement was left to be read from the context. Part C in the figure shows the use of division-like algorithm to find the square root of 8 and part D is jottings made while trying to find a number twice the square of which is a square as well. The only part of writing that would be acceptable in a school assessment is the square root algorithm. All the parts though supported their mathematical thinking.
5) Also worth noting are the positive feelings implied in the statement “we found out ourselves”, the confidence with which they pointed to the teacher that a square has a right angle, and the clapping that the finding out occasioned.

Examining the discourse for the features that Sfard (2008) describes as characterising mathematical discourse we see that the discourse here deviates widely. Word use is not objectivised. The attempt here is on fitting the matchstick diagonal and Pythagoras theorem is stated with respect to the triangle at hand rather than in a context independent, objective manner. The conversation is visually mediated by matchsticks and diagrammatic representations as opposed to symbols and abstractions. The aim here is to come up with statements regarding fitting a diagonal to a matchstick square. The students are rooted in the concrete manipulatives and are not making claims about abstract squares. However, at least some of them use deductive reasoning to endorse narratives. They argue for the impossibility of fitting the diagonal using the general statement that the diagonal of a square is "always" greater than its side. This may not have led to the expected conclusion but there was an attempt to argue deductively. In fact all three of deductive arguments, inductively drawn conclusions and conclusions made on the basis of practical action were seen, giving rise to questions about what students think about the validity of these means of endorsing narratives in mathematics.

In contrast to Sfard, Moschkovich (2015) does not insist on the use of specialised vocabulary or symbols to classify a discourse as mathematical. She suggests that mathematical discourse is embedded in practices and draws on hybrid resources and multiple registers. She also marks such features as precision, brevity, logical coherence, particular modes of argument and tendency to value abstraction, generalisation and search for certainty, not all of which can be found in the exemplified instance. A focus on how the discourse differs from the expected characteristics of mathematical discourse may lead to a deficit perspective that fails to acknowledge the mathematical in such discourses. Though the discourse itself deviates from what is considered mathematical discourse in literature, there are "family resemblances" it shares with mathematical discourses and definitely show elements of mathematical thinking, though expressed in deviant ways. This brings home the need for more flexible acceptability criteria for mathematical discourse that focuses on the resemblances that they bear to mathematical discourse.

4.2.3 How does language support or hinder mathematical communication?

Our study also corroborates the findings of other scholars (Barwell, 2016; Bose & Choudhury, 2010; Moschkovich, 2008; Setati et al., 2002) that the simultaneous presence of the home language and the school language, formal and informal mathematical language supports students in expressing mathematical ideas meaningfully. However we also have instances where language has been a hindrance to further progress.

In the instance shared, the student S1 is taken aback by the teacher's question of how much bigger and does not immediately respond to the question. The language necessary to express a multiplicative relationship could have been the stumbling block. The relation between the side-length and the diagonal of the square was expressed in less precise terms as "not equal" and "greater". Being able to express it multiplicatively would have made the solution to the problem obvious and allowed for generalisation as well. This is an instance where less precise and informal language hindered progress.

On a similar note, Pythagoras theorem gets mentioned when the students argue for the impossibility of the diagonal of a unit square, but was not articulated precisely or in symbolic form. Even when the symbolic form was mentioned in the context of rectangles for which diagonals could be fitted, the students did not clarify what the symbols meant. Their statement that the diagonals could be fitted only for those rectangles for which $a^2 + b^2 = c^2$ indicates an incomplete understanding of the equation and the range of possible values the variables a, b, and c could take. Having mentioned Pythagoras theorem in one specific case, the realisation that the same holds for another specific case (that of square of side 2) was not immediately obvious to them. When looking for squares within which a matchstick diagonal could be fitted, their approach was to examine specific cases. No amount of trying out will yield a positive example here, nor does not being able to find an example prove anything and further progress in the problem is blocked. Had they formalsied the problem as for a given x, finding y such that $2x^2 = y^2$ the general solution would have been within reach. The lack of formalisation proved to be a stumbling block to solving the problem. We have had similar observations in other modules as well, and infer that while informal language helps to get started on an exploration, lack of formal language may hinder progress beyond the initial stages.

4.2.4 What counts as mathematical discourse in such contexts?

We saw that Sfard’s (2008) characterisation of mathematical discourse including such features as objectified word use, visual mediation by symbolic artefacts, routines that result in endorsable mathematical narratives and deductively endorsed narratives as being too stringent a characterisation to be inclusive of the observed discourse at the margins. Moschkovich (2015) is accepting of the vernacular in academic contexts, of multiple modes of communication as long as there is evidence of mathematical processes. While highlighting the need to start from where the learners are and using their language as resource, the Academic Literacy in Mathematics (ALM) framework relegates formalisation to a subsidiary role. While the formal discourse of the textbook could be hard to navigate for learners, learning it is advantageous to them. For one, as observed in the previous section, formalisation is an enabling factor for generalisations and extensions of the problem. Also, it is valued in the practice of mathematics as a means to build coherence and catch contradictions when they arise. Looking beyond school mathematics, formal language becomes necessary to gain access to higher mathematics and to the educational and social benefits that this entails. So we need an acceptability criterion for students’ informal discourses that keeps formalisation in view as well.

While research mathematicians use pictures and highly ambiguous terminology and notation during discussion and discovery, they proceed to use a more formal language in writing definitions, assertions and their proofs, mainly to discover possible inconsistencies, or possible gaps in proofs (Barwell, 2013; Hadamard, 1945). Mueller-Hill (2013) identifies formalisability as one of the epistemic features of proofs in research mathematics. She goes on to suggest that formalisability of discursive proving actions, be seen as a metadiscursive rule, in the sense of Sfard (2008). Keeping in view the epistemic features of mathematics while being accepting of student contributions, we propose "*coherent formalisability*" as an acceptability criterion for mathematical discourses (Jayasree et al., 2023). Informal discourse is formalisable if, given sufficient additional mathematical resources like missing terminology, definitions and reasoning as support, it can be restated in formal terms. We expect some structural similarity to a formal discourse, though not all elements of the structure may be present explicitly. There may be multiple ways to augment the discourse and map it on to a formal one. However, such a criterion could be trivial at the level of individual statements or assertions; hence the qualifier of coherence, that binds statements together into a whole.

In the episode described above, the students argued that placing one stick along the diagonal left gaps and that there was not enough space to place two sticks and therefore a matchstick diagonal cannot be fitted without gaps and overlaps. The warrant that the students use in justifying their conclusion is based on practical reason and thus deviates from Sfard's characterisation of mathematical discourse. However, this can be formalised as d the length of the diagonal is > 1 and < 2 and is therefore not an integral multiple of the side length, 1. While it is formalisable, it is highly contextual and applies only to the case of a unit square. To argue for the general case, the students draw on the property that the diagonal of a square is greater than the sides. By itself this does not lead to a valid argument. The teacher's nudge towards the missing information that diagonal is not just "greater" than the side-length, but an irrational multiple of side-length, can be seen as a nudge towards a formalisable argument.

This leads to questions about the role of the teacher in facilitating explorations at the margins, being flexible about what counts as mathematical discourse while at the same time helping students move towards more formalised discourse.

4.3 Role of teacher in balancing flexibility and disciplinary considerations

The goal of our study was to examine the potential of explorations to create affordances to engage with mathematics in ways that differ from the preferred ways of the curriculum and textbooks. The flexibility afforded by explorations also means additional challenges for teachers. Based on my reflections on my own teaching and the challenges I faced in facilitating explorations, we sought to identify the additional challenges that explorations bring and suggest ways of addressing them.

Absence of ready-to-use reference material: We are not aware of many ready-to-use reference material for explorations that parallels a textbook for curricular content. A library-search or internet-search yields a number of potential starting points or articles or papers that can be framed as explorations. The actual design of the exploration, anticipating possible variations and trajectories along which the explorations could branch out, requires the teacher to engage with the task as an explorer herself and requires considerable time and effort.

Need for deep content knowledge: An exploration involves much more than solving the initial problem posed. It is likely that the exploration progresses to questions to which the teacher does not have an answer or is yet unanswered by the community of mathematicians. The multiple trajectories and approaches possible within an exploration also place additional knowledge demands on the teacher.

Need to recognise mathematics in students' contributions: Students are engaged in the process of discovering things for themselves as they engage in explorations. It is likely their insights are articulated in half-formed informal or semi-formal terms. The approach that the students take may not be what the teacher anticipated or even be familiar with. So to listen at the margins, recognise the formalisation implied in an informal expression and nudge the students accordingly could be a challenging task.

In the thesis, we suggest guidemaps for explorations prepared by seasoned explorers of mathematics and/or research mathematicians, as means to mitigate these challenges and suggest a framework for such guidemaps.

A flexible acceptability criterion for expressing mathematics implies that the teacher has to deal with unfamiliar language in addition to unfamiliar mathematics. In a situation where the teacher’s socio-cultural background differs markedly from that of students, the teacher needs to be sensitive to her own biases coming from her background and where she is *listening from* (B. Davis, 1994). Non-standard terminology that students use, incomplete or inappropriate articulation, hidden assumptions and intentional hedging and vagueness that students bring in when they are uncertain, all pose challenges in terms of language. Sometimes the informal ways of articulation that the students resort to may require a lot of gap-filling for the teacher to make sense of. The teacher needs to go beyond hearing and interpreting the students' nascent formulations to responding to the underlying thinking, what Davis (1997) terms as hermeneutic listening. The teacher needs to listen for leverage points in students' work and be sensitive to potential violations of coherent formalisability. We discuss details in the thesis.

4.4 What does mathematical engagement look like in curricular contexts?

Though the study was not designed to address this question, the textbook based lessons that I taught at the school's request gave us some pointers to what mathematical engagement might look like in curricular contexts. Some of the sessions were explicitly meant as practice sessions to solve textbook exercises. However we noted students responding in ways which we did not expect. We highlight some aspects that we noticed.

Focus on sense-making rather than "following the procedure: An often identified aspect of the school maths tradition is the tendency to replicate solution procedures and algorithms not necessarily knowing why. Markedly different from this, we noted students questioning some met-before and taken-for-granted concepts and algorithms in some of the classes meant to "solve revision problems" towards the end of the two-year teaching experiment. Students questioning the rationale for the step of doubling the number currently "at the top'' while performing the algorithm for finding the square-root and their attempt at making sense of multiplication of irrational numbers within the scheme of the usual multiplication algorithms are illustrative instances which we describe in the thesis.

Shift from obtaining the answer by a given method to appreciating and adopting alternate methods: The end of section exercises in textbooks are intended to solve problems related to the concept learnt in the section. The textbook also has "worked out examples" whose methods of solution are taken as preferred methods. The assessment criteria are such that there are no special credits for more elegant solutions. However we noted instances where students appreciated alternate solution strategies, remarked on the simplicity of the approach compared to that taught in regular class and were keen to adopt it for subsequent problems. We see in this a shift of focus from the "right answer" and a developing sensitivity to alternate approaches to the solution.

Aspirations and willingness to take on challenges: We also noted instances of some students asking to be given more difficult problems and encouraging others to engage with such problems. Though the class was meant for exam preparation, they did not want to limit themselves to direct problems involving substitution in formulae. We also had a glimpse of their aspirations beyond passing the exam when a student said "*nariya padikkanum. Ungala mathiri IMSckku pokanum*" (I want to study a lot. I want to go to IMSc like you) and asked for guidance.

Agency: A very encouraging feature that we observed was the agency that the students were willing to assume, be it in volunteering to be the "teacher for the day" and assigning problems to the rest of the class, or in asserting their own choice of methods different from the one preferred by the teacher or taking part and helping in the research process by handling the audio recorders and offering to be observers.

Though not making any causal claims we hypothesise that the flexible pedagogy adopted, supported by the flexibility afforded by explorations and inquiry stance that was more relationship building than surveillance oriented (Vossoughi & Escudé, 2016) might have had a part to play in bringing about these observed differences from the expectations.

5. Conclusions, Limitations and Further Work

5.1 Conclusions and implications

Starting from the well acknowledged fact that mathematics contributes to marginalisation of some students, we identified three dimensions - performance dimension, disciplinary dimension and language dimension - along which this happens. We sought to "recentre" the disciplinary dimension of the margin by creating accessible tasks that allow for engagement with mathematics more flexibly than afforded by the curricular context. We observed students engaging with these tasks. They engaged in such practices of mathematics as conjecturing and convincing, specialising and generalising, representing and rerepresenting, imagining and expressing. However we noticed that talk was their preferred way of expressing their mathematical thinking. They overcame the limited access they had to formal and written mathematical language by drawing on multiple resources - self-created terms, everyday/informal language, gestures to express themselves. Accepting these 'atypical' ways of communication and "recentering" the language dimension of the margin meant defining more accommodating criteria for what constitutes mathematical discourse. Keeping the core aspect of formalisation and acknowledging its role in ensuring consistency in view, while not insisting on the formal, we suggested *coherent formalisability* as such a criterion. We also noted desirable student behaviour such as trying to make sense of what they were doing as opposed to replicating procedures, assuming agency and aspiring for excellence. We conclude that well chosen explorations and privileging talk anable flexibility and access to mathematics at the margins and are therefore means to recenter the margin.

Though the study itself was done in a school catering to students from socio-economically disadvantaged backrounds, we suggest that the conclusion also applies to 'mathematically marginalised students' from any background. Based on this the study has the following implcations.

- Curriculum needs to be reorganised to allot time and space for explorations and talk. These should be prioritised and not held hostage to board exams.
- Assessment needs to go beyond those relying entirely on writing and should factor in talk as well
- Teacher education needs to include elements that enable teachers to

implement a flexible pedagogy

We draw attention to the tendency in maths education to acknowledge the equity dimension, but to accept and internalise the deficit perspective. An important marker of deficit perspective are statements that focus on students' academic and intellectual shortcomings, with little or no recognition of their existing understandings and strengths. Building on scholars who suggest an anti-deficit noticing as means to recentre the margin, we further suggest reinterpreting/reframing deficit using a metaphor of "distance" rather than a "gap" and suggest that this reframing recasts distance as something that is transient or variable, and would be traversed with passage of time. In place of the "deficit - anti-deficit" binary, distance provides a spectrum, the possibility that a teacher who has a deficit perspective may yet listen and alter her perspective. Accordingly, "recentering" the margin would entail negotiating and traversing this "distance", We suggest that distances could be framed in deficit terms, leading to the foregrounding of "deficit distances", or alternatively in non-deficit terms, leading to noticing and giving importance to "potential distances". An example of "deficit distance" is the gap between grade level expectation of mathematical knowledge and students' knowledge as elicited through examinations. In contrast, the distance between what students are mathematically capable of and what is acknowledged by the institution (teachers, schools, exams, etc.), would be an example of "potential distance". Our attempt in the instances described in the preceding sections has been to highlight potential distances.

Recentering the margin would require commitment and action at a larger systemic level along with the pedagogic changes we have discussed. We hope that this study contributes some initial steps in this direction.

5.2 Limitations and future work

This study focussed on student mathematical thinking at the margins and ways that they express their thinking and the influence of flexibility on their mathematical engagement. The role of the teacher in enabling this flexibility has not been well-researched. The insights that we offer on the challenges a teacher might have in this and the suggested workaround of guidemaps are based on the reflections and experience of the research team. We need to work further with teachers to gain deeper understanding of both the usefulness of our guidemaps and teachers' resourcefulness in using them effectively.

In addition to guidemaps, there is also a need to develop assessment rubrics for explorations. What does it mean to "progress" in an exploration? What are pointers that one can look for to make sure that students are making mathematical gains, for instance, progressively moving towards more formal means of communication. Such rubrics are important both from the perspective of student evaluation and providing teacher support to facilitate explorations.

Writing is an important aspect of learning mathematics. Our primary focus has been on talk and we relegated writing to the background in the interest of keeping up student engagement. However there is a need to study the nature of writing that an exploratory context calls for, the kind of writing that students produce in such contexts and what may be considered acceptable writing. Further the differences between written and spoken natural language (such as in Tamil) may also have an impact on students writing mathematics.

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